University of Groningen
Exam Numerical Mathematics 1, June 19, 2017

Use of a simple calculator is allowed. All answers need to be motivated.
In front of each question you find a weight, which gives the number of tenths that can be gained in the final mark. The maximum total score for this exam is 5.4 points.

Exercise 1

(a) Let $n+1$ points (x_i, y_i), $i = 0, 1, ..., n$, be given with distinct nodes x_i. A polynomial P is called interpolating if $P(x_i) = y_i$, $i = 0, 1, ..., n$.

- (i) $\boxed{4}$ Give a complete description of the Lagrange interpolation formula, and explain why this formula provides an interpolating polynomial P of degree $\leq n$.
- (ii) $\boxed{2}$ Show that there cannot exist another interpolating polynomial P of degree $\leq n$.
- (iii) $\boxed{4}$ Suppose that all the nodes x_i lie in an interval $I = [a, b]$, and that we are interested in evaluating the interpolant P at arbitrary $x \in I$. How is the corresponding Lebesgue constant Λ defined, and what are the implications if its value is large (say, $\Lambda = 10^5$) ?

(b) For a smooth function f on the interval $[0, 1]$ we approximate its (one-sided) derivative $f'(0)$ by $P'(0)$, where P is the polynomial (of degree ≤ 2) that interpolates f at the nodes $x_0 = 0$, $x_1 = h$ and $x_2 = 2h$.

- (i) $\boxed{1}$ Show that P is given by

$$P(x) = \frac{f(0)}{2h^2}(x-h)(x-2h) - \frac{f(h)}{h^2}x(x-2h) + \frac{f(2h)}{2h^2}x(x-h).$$

- (ii) $\boxed{1}$ Use the above explicit expression for $P(x)$ to show that

$$P'(0) = \frac{1}{2h}\left[-3f(0) + 4f(h) - f(2h)\right].$$

- (iii) $\boxed{3}$ Show that $P'(0)$ is a second order approximation of $f'(0)$ (with respect to h).

Exercise 2

(a) Consider a system of nonlinear equations $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

- (i) $\boxed{4}$ Derive Newton's method for the above system, and explain briefly how this method works.
- (ii) $\boxed{3}$ Consider the above system with $n = 2$ and

$$f_1(x_1, x_2) = x_1 + x_2^2 + \sin(x_1 x_2) - 3, \quad f_2(x_1, x_2) = x_1 + x_2 + \cos(x_1 x_2) - 4.$$

Starting from the initial guess $x^{(0)} = (\pi, 0)^T$, show that Newton's method converges to the root $\alpha = (3, 0)^T$ in a single step.

(b) Consider the fixed point iteration $x^{(k+1)} = \phi(x^{(k)})$ with $x^{(0)}$ given and $\phi(x) = \frac{1}{3}x(4+x-2x^2)$.

- (i) $\boxed{1}$ Determine all fixed points α of ϕ.
- (ii) $\boxed{4}$ For each of these fixed points α, check whether $\{x^{(k)}\}$ converges to α if $x^{(0)}$ is chosen sufficiently close to α. If that occurs, also determine the order of convergence.

Continue on other side!

Exercise 3

(a) Consider the system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 9 \\ 10 \end{bmatrix}.$$

(i) [4] Determine the Cholesky factorization and LU factorization of A.

(ii) [3] Use one of these factorizations to solve $Ax = b$.

(b) For solving a general linear system $Ax = b$ we consider iterative methods of the form

$$Px^{(k+1)} = (P - A)x^{(k)} + b,$$

where P is a nonsingular preconditioner of A.

(i) [1] Determine the iteration matrix B and show that the error $e^{(k)} = x^{(k)} - x$ satisfies $e^{(k+1)} = Be^{(k)}$. When does the method converge?

(ii) [5] What is the name of the iterative method that corresponds to the preconditioner $P = D = \text{diag}(a_{11}, a_{22}, ..., a_{nn})$? Show that this method converges if A is strictly diagonally dominant by row.

Exercise 4

(a) For the numerical solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

we use the so-called *implicit midpoint rule*, which is defined as

$$u_{n+1} = u_n + hf(\frac{1}{2}t_n + \frac{1}{2}t_{n+1}, \frac{1}{2}u_n + \frac{1}{2}u_{n+1}),$$

where $u_0 = y_0$ and $t_n = t_0 + nh$.

(i) [3] Show that application of this method to the test problem $y'(t) = \lambda(t)y(t)$ leads to the recurrence relation

$$u_{n+1} = \frac{1 + \frac{1}{2}h\lambda(\frac{1}{2}t_n + \frac{1}{2}t_{n+1})}{1 - \frac{1}{2}h\lambda(\frac{1}{2}t_n + \frac{1}{2}t_{n+1})}u_n.$$

(ii) [4] Give the definition of 'A-stability' (unconditional absolute stability) and verify whether the implicit midpoint rule is A-stable.

(b) Consider the Poisson equation on the (open) unit square $\Omega = (0, 1) \times (0, 1)$,

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

where $u(x, y) = g(x, y)$ is given on the boundary of Ω (Dirichlet boundary conditions).

(i) [2] First show that for any smooth function $v : [0, 1] \to \mathbb{R}$ and $x \in (0, 1)$ the quantity

$$\frac{v(x + h) - 2v(x) + v(x - h)}{h^2} \tag{2}$$

provides an approximation to $v''(x)$ of order 2 with respect to h.

(ii) [5] We choose an integer $N \geq 1$, set $h = 1/(N + 1)$ and define grid nodes $(x_i, y_j) = (ih, jh)$, $i, j = 0, 1, ..., N+1$. We construct approximations $u_{i,j}$ to $u(x_i, y_j)$ by requiring that differential equation (1) is satisfied at all internal grid nodes while replacing both second derivatives by the second order difference quotient of type (2). This leads to a linear system $A\tilde{u} = b$, where the vector $\tilde{u}$ consists of all values $u_{i,j}$ at the internal nodes. Find the matrix A and right-hand-side vector b in case $N = 3$.